Learning Trajectories and Rational Number Reasoning

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Workshop on Higher Cognition in Adolescents and Young Adults:
Social, Behavioral, and Biological Influences on Learning
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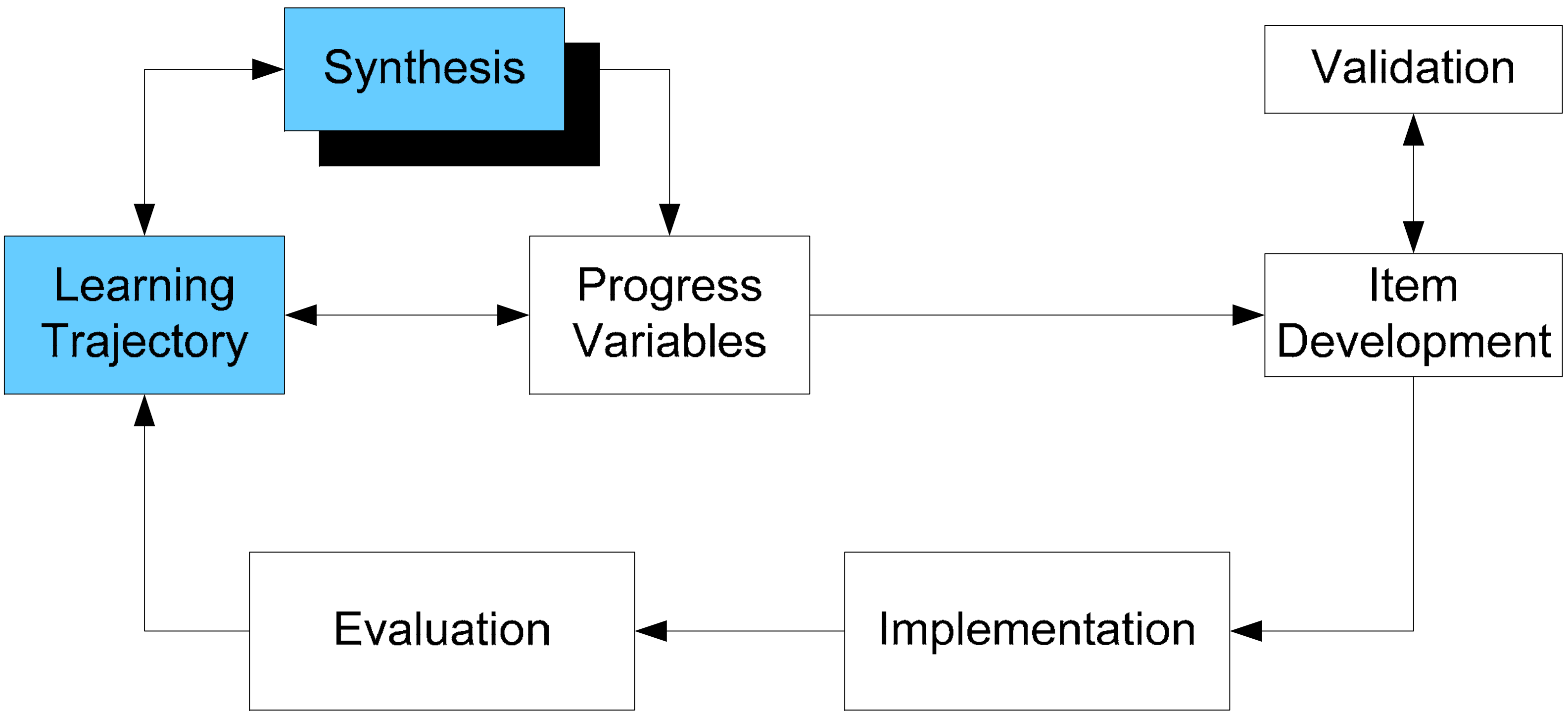
Metrics of Higher Cognitive Thought

- Better measurement of high-order cognitive processes, particularly those that develop gradually, represents a key problem.

Part of a Larger Study to Build Diagnostic Measures (Diagnostic E-Learning Trajectories Approach--DELTA)



Better understanding of the higher-order cognitive processes requires careful attention to:

- the set of related mathematical ideas
- the forms of reasoning characteristic of the discipline
- the evolution of learning in relation to student thinking or development of learning trajectories.

Rational Number Reasoning Synthesis
(625 papers)

1. Equipartitioning/Splitting (34)
2. Multiplication and division (116)
3. Fractions (195)
4. Ratio, proportion and rate (142)
5. Area and volume (64)
6. Similarity and scaling (12)
7. Decimals and percents (62)

Database

- title
- author
- source type
- theoretical/empirical nature of study
- topic
- grade level
- assessment items
- the study demographics
- analysis
- (new) abstract

Synthesis

- “The investigator must propose overarching schemes that help make sense of many related but not identical studies.” *(p. 12*).
- “The cumulative results are more complex than any single study, because they have to explain higher-order relations.” (p. 13)
- “Perhaps the most challenging circumstance in the social sciences occurs when a new concept is introduced to explain old findings.” (p. 17)

Cooper (1998), *Synthesizing Research*

Rational Number Research

- Rational Number Project: fraction, decimal, ratio, indicated division, measure and operator. (http://cehd.umn.edu/rationalnumberproject/)
- Multiplicative Conceptual Field:
  - mathematical framework embedding conceptual operations, the situations that tie to the child’s experience, a “bulk” of concepts, symbolic and linguistic signifiers.
  - multiplication and division, linear and bilinear (an *n*-linear) functions, ratio, rate, fractions and rational numbers, dimensional analysis, linear mapping and linear combination of magnitudes. (Vergnaud, 1983, 1988, 1994)
- Fractions as extensions of whole-number reasoning on the number line (NMAP)

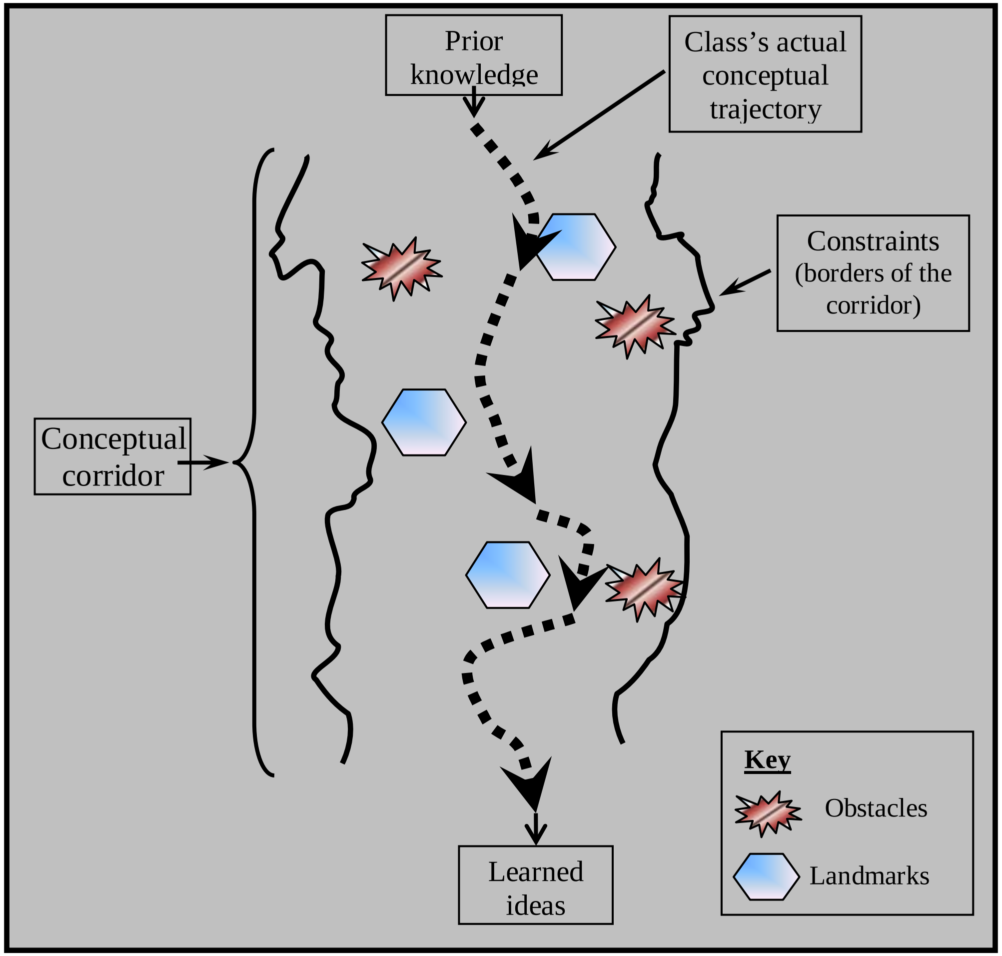
Defining a learning trajectory

Simon (1995), Cobb *et al*. (2003), Confrey (2006), Lehrer and Schauble (2006)

- Learning Trajectory:

  A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.

- Confrey *et al*. (2007)



Confrey (2006) Design Studies Chapter
Cambridge Handbook of the Learning Sciences

Two parts of presentation:

- Summary of Key Findings of Synthesis
- Update on development of measures

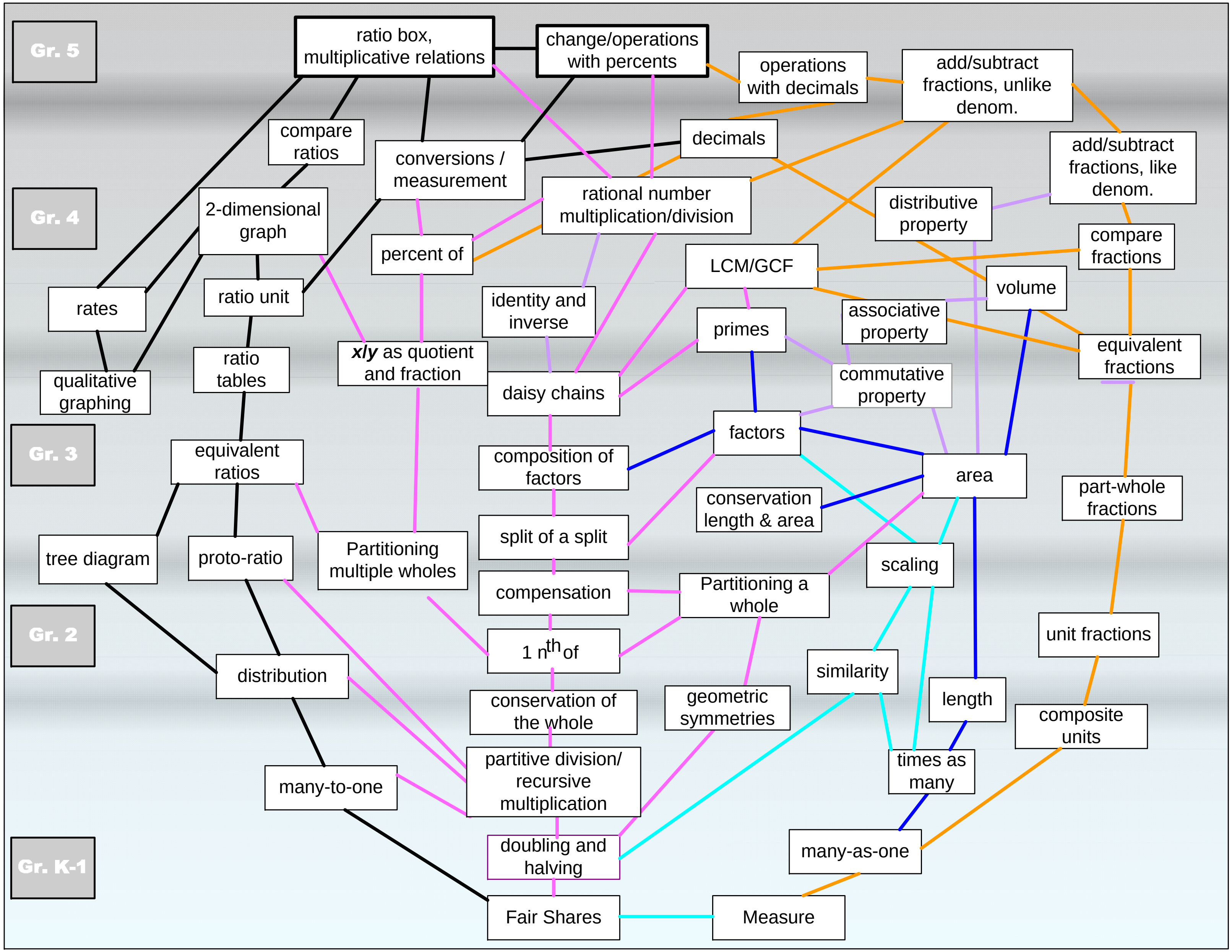
Key Findings:

- Rational Number Reasoning is complex, and yields to a Learning Trajectories strand analysis;
- Equipartitioning/Splitting is the foundation for Rational Number Reasoning;
- Division and multiplication should be derived from equipartitioning/splitting, and coordinated with counting, addition, and subtraction;
- Three dominant meanings for *a*/*b* capture most of RNR reasoning; and
- Fundamental revisions are needed in the sequencing of RNR construct learning.

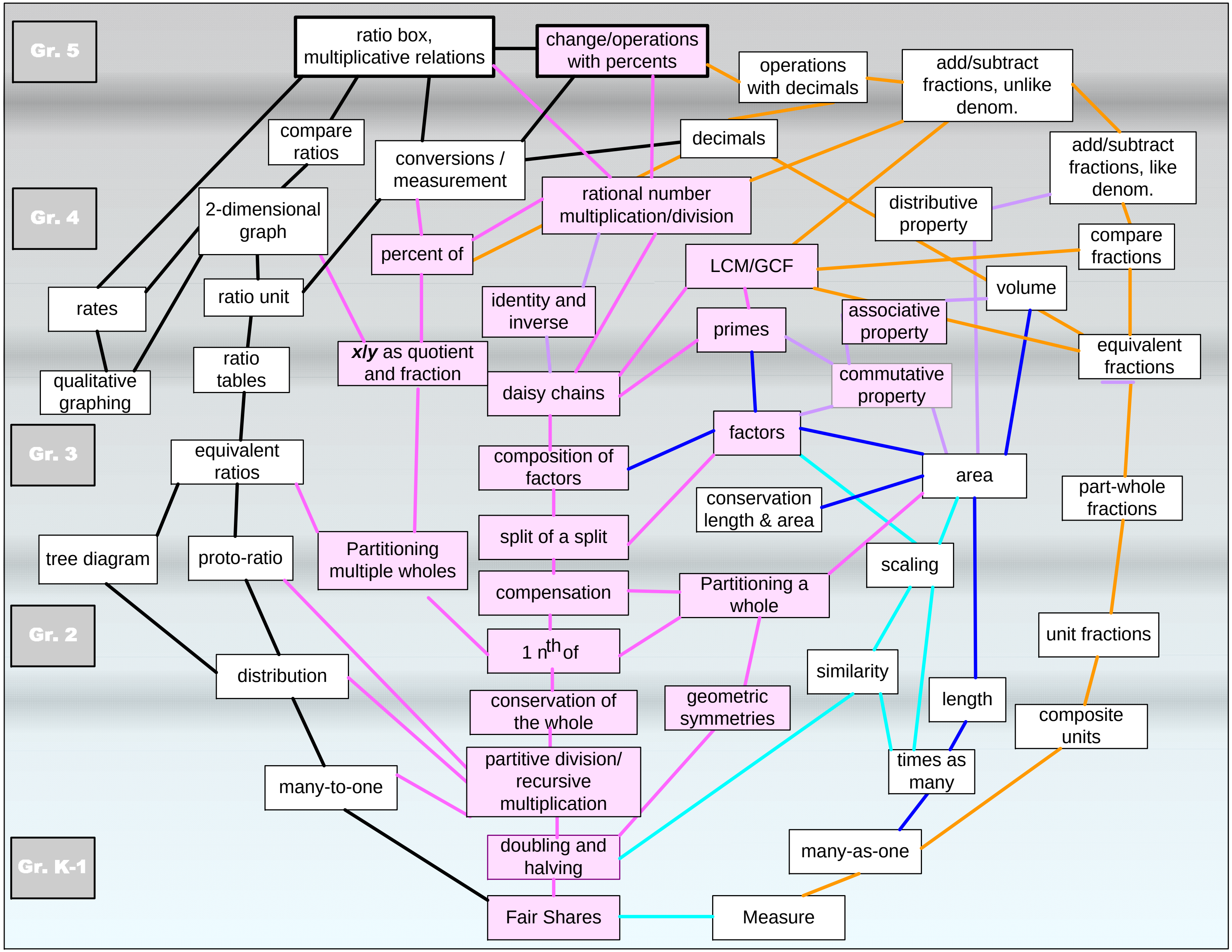
Key Finding 1:
Complexity and Learning Trajectories

- Rational Number Reasoning is complex, and yields to a Learning Trajectories strand analysis.

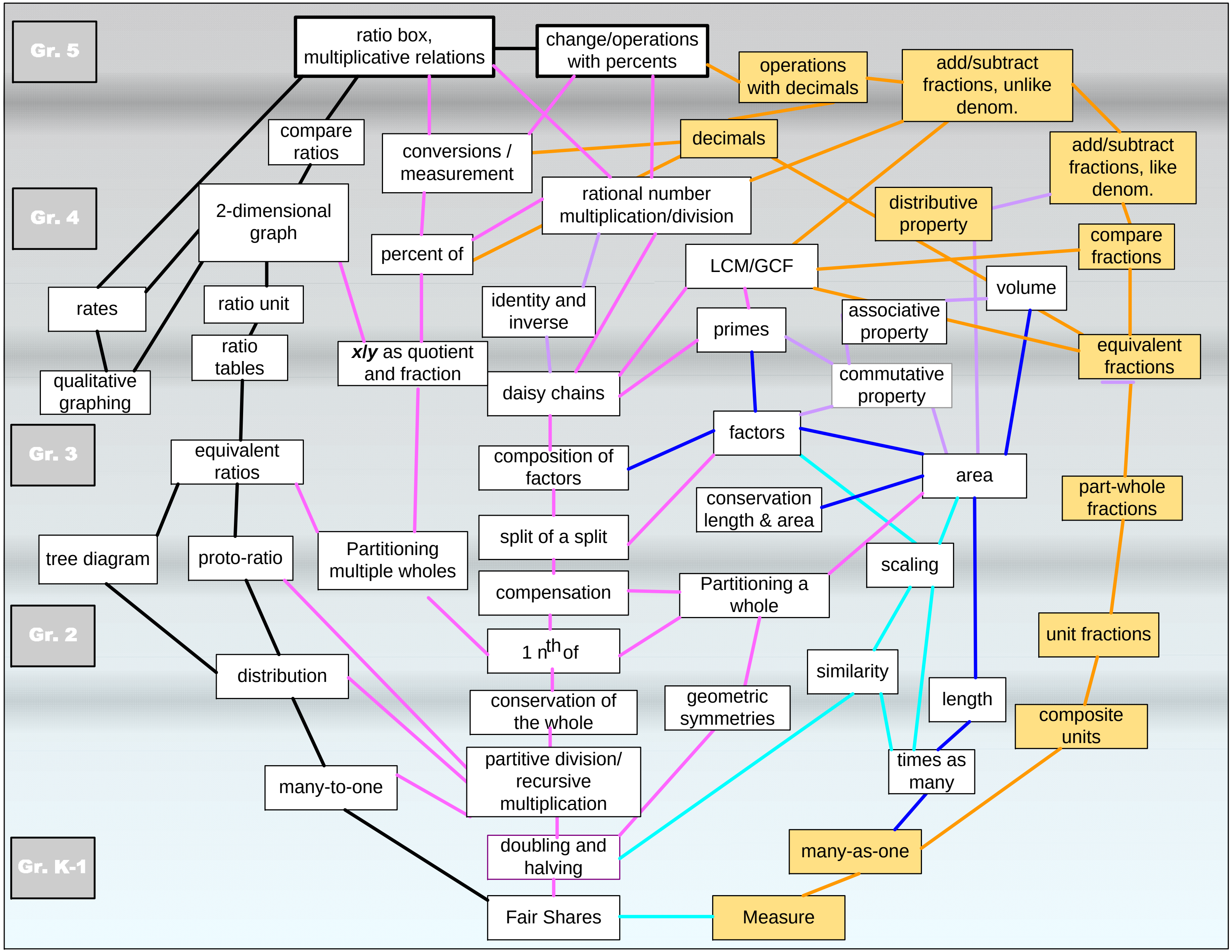
RNR Learning Trajectories



Partitioning to Multiplication/Division



Fractions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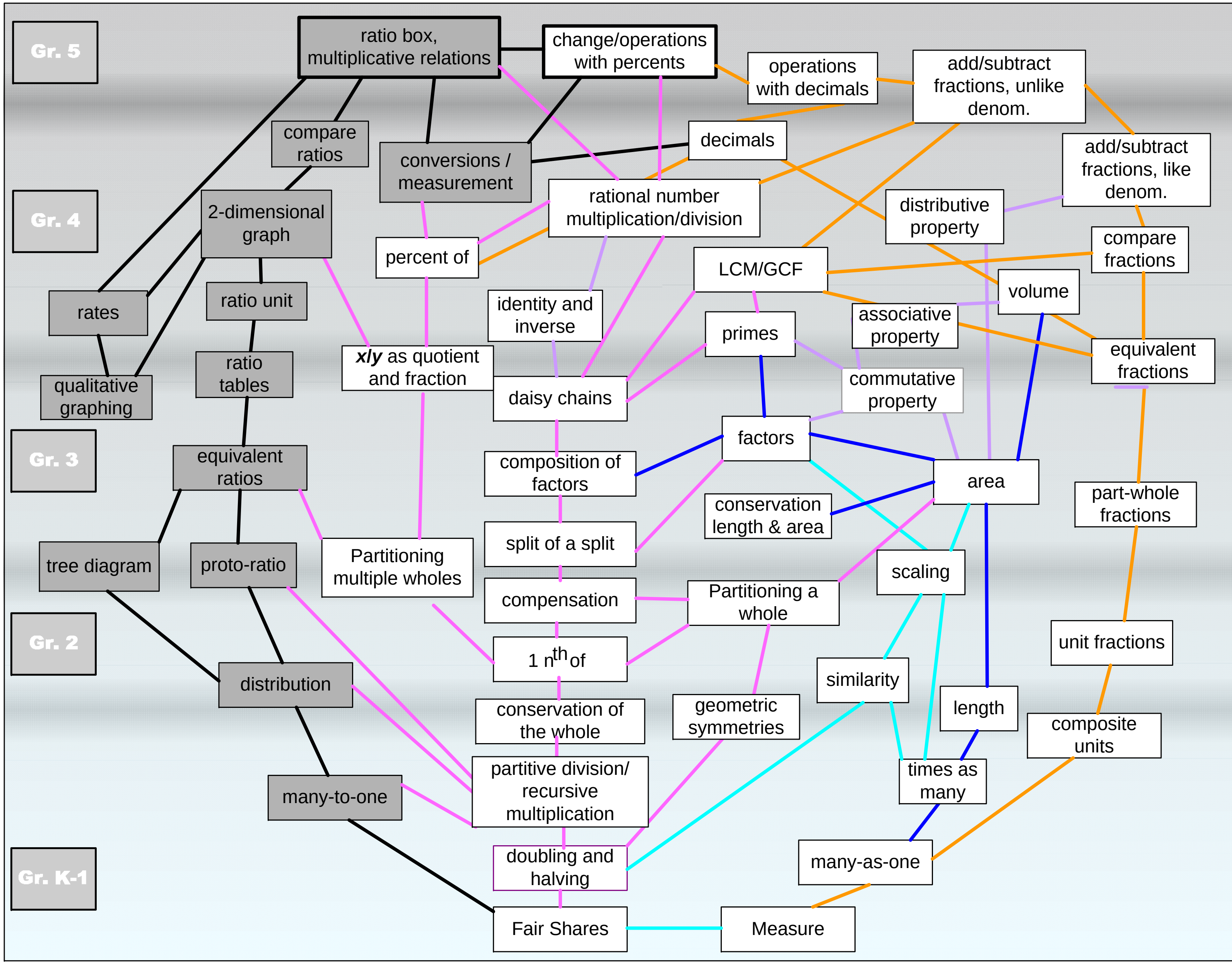
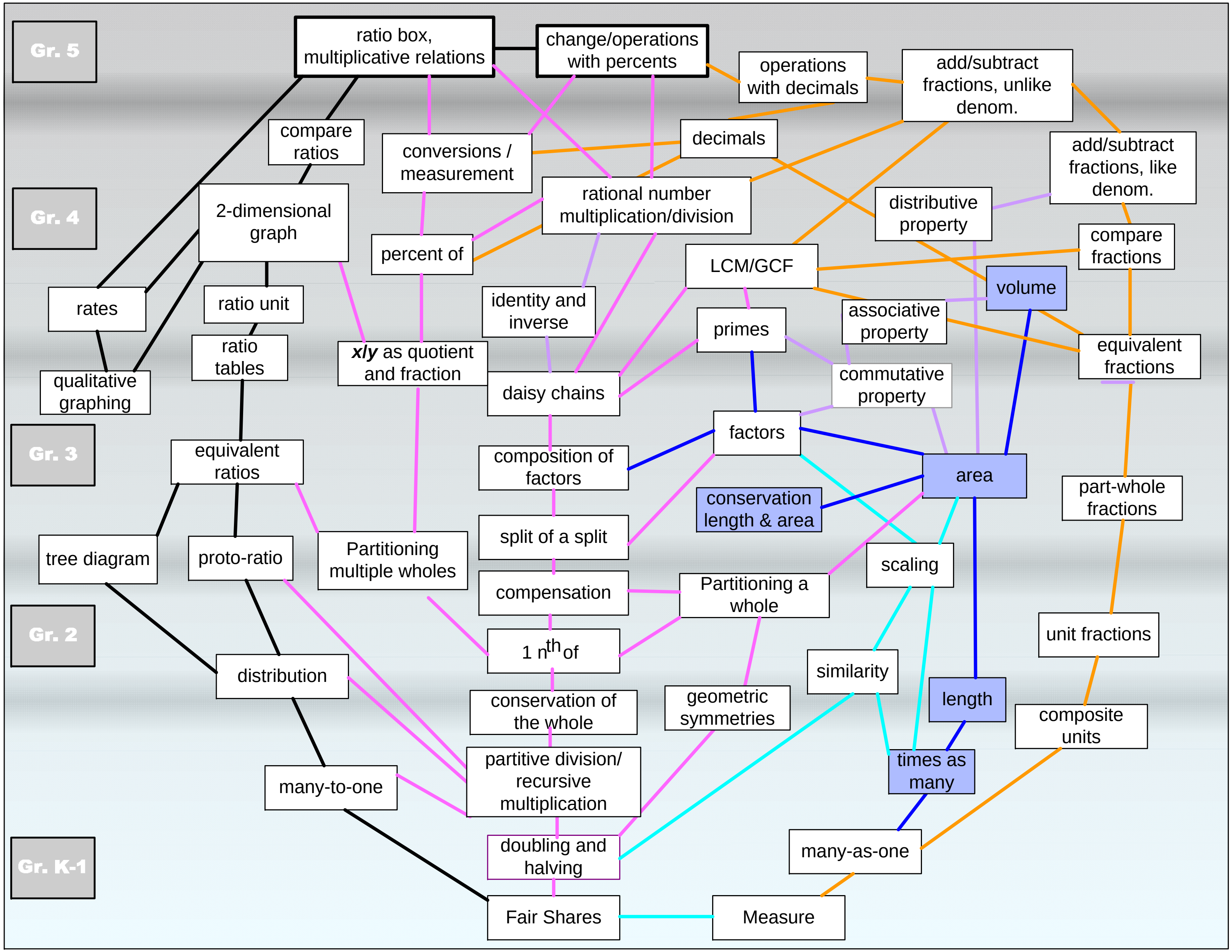
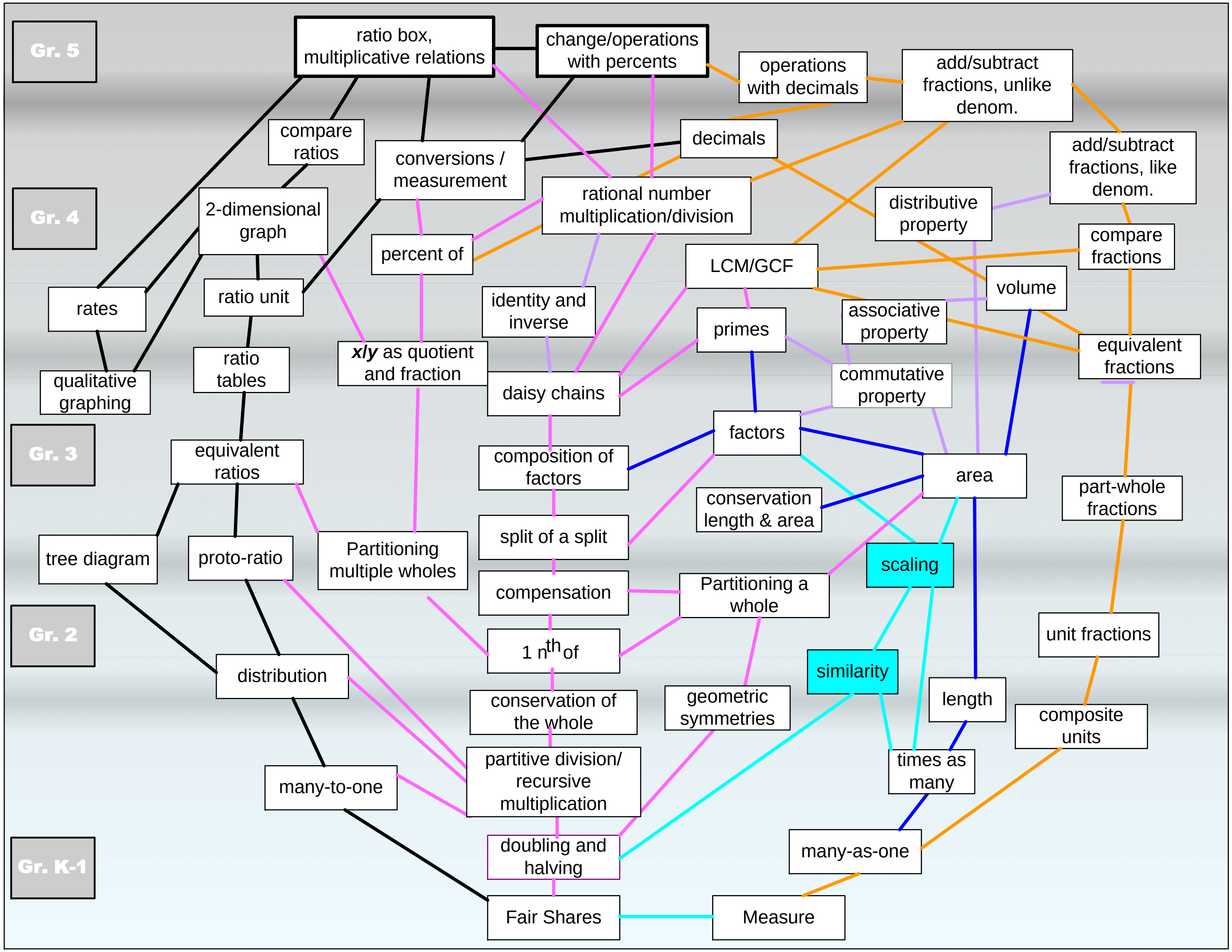
Ratio, Proportion, Rate



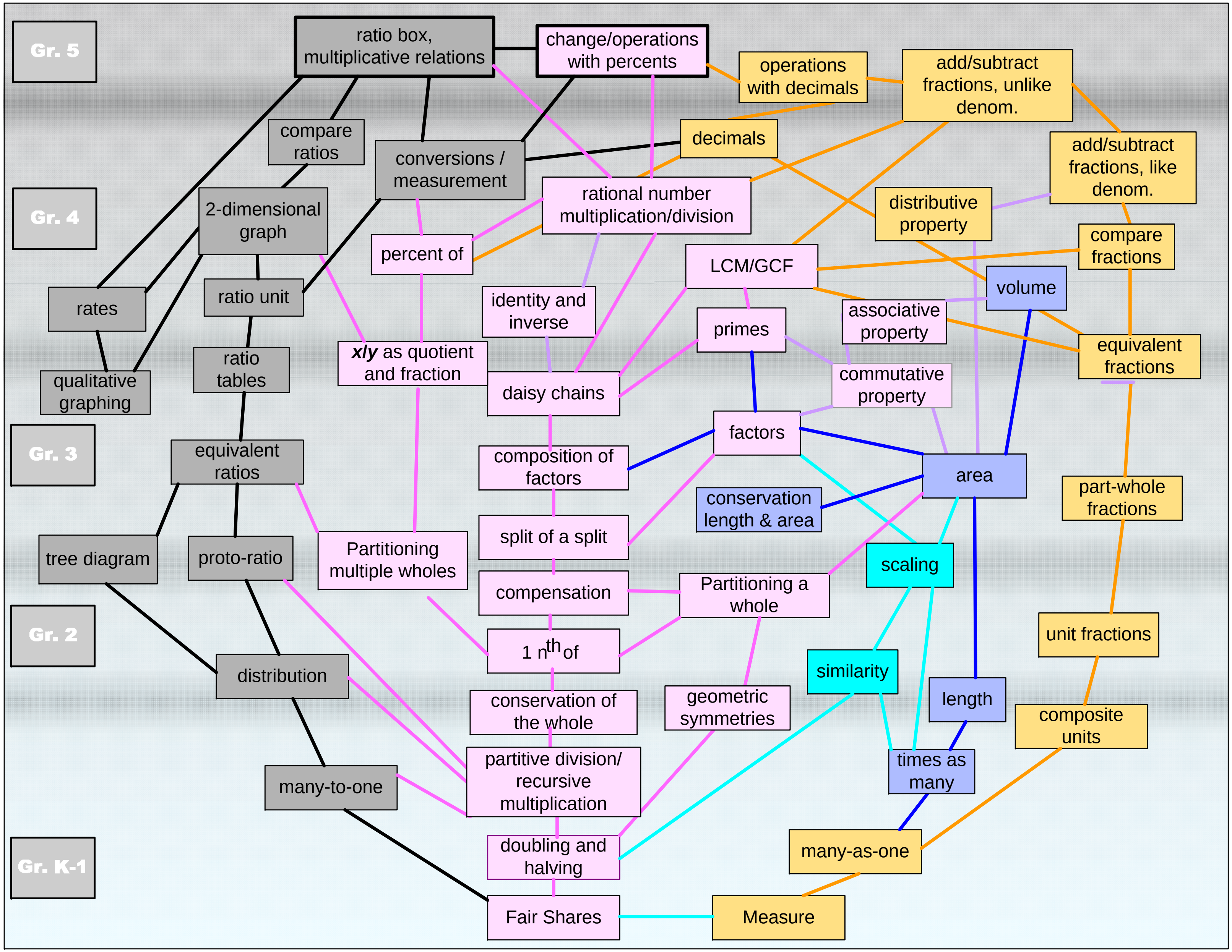
Area and Volume



Similarity and Scaling



RNR Learning Trajectories



Summary of Finding 1: Why a learning trajectories approach?

- Permits us to respect complexity yet disentangle it;
- Permits us to build from the cognitive resources children bring to school from informal settings;
- Recognizes that the “logical structure of mathematics” and cognitive development in mathematics are not identical; and
- Permits us to view expertise as refinement of approach over time.

Key Finding 2

- Equipartitioning/Splitting is the foundation for Rational Number Reasoning;

Defining Equipartitioning/Splitting

- Equipartitioning/Splitting indicates cognitive behaviors that have the goal of producing equal-sized groups (from collections) or pieces (from continuous wholes) as "fair shares" for each of a set of individuals.

- Equipartitioning/Splitting is not breaking, fracturing, fragmenting, or segmenting in which there is the creation of unequal parts.

- Equipartitioning/Splitting is the foundation of division and multiplication, ratio, rate, and fraction.

Defining Cases of Equipartitioning/Splitting

- Case A: 15 coins among 3 pirates
- Case B: 1 cake among 4 people
- Case C: 3 cakes among 4 people
- Case D: 5 cakes among 4 people

Common Themes

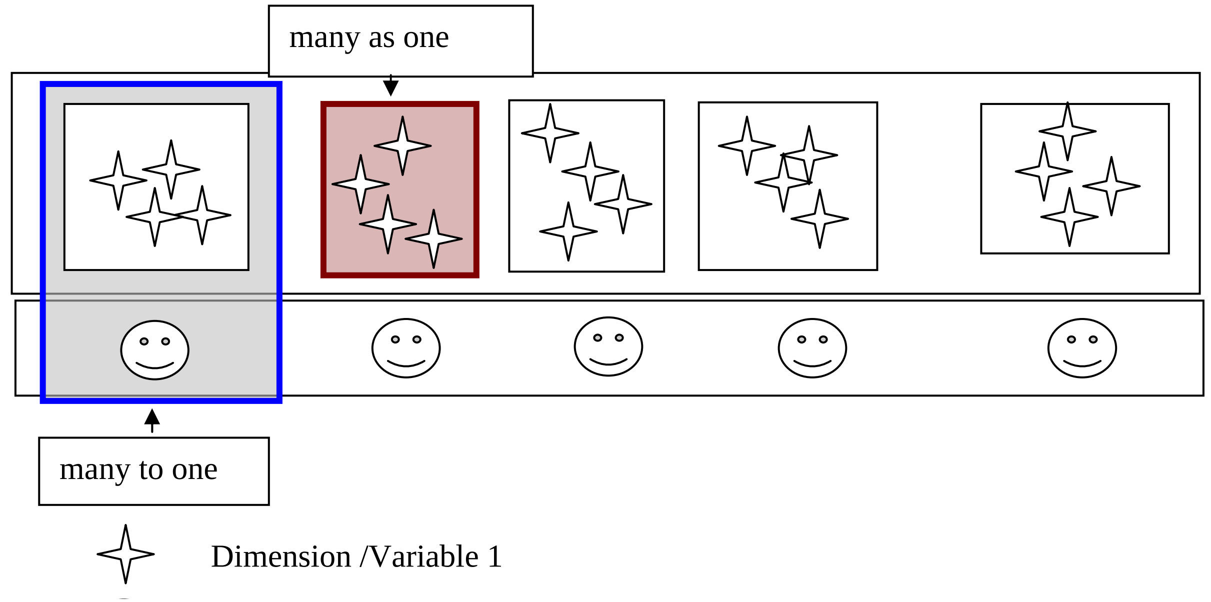
- Students highly successful at young ages
- Strong connections to number-theoretic and geometry properties
- Equipartitioning/Splitting as the source of division
- “Fair Shares” can be a basis for unit ratios and unit fractions

Students highly successful at young ages

- Ex.: Pepper (1991)
  - 96 pre-schoolers
  - 12 cookies, 2 dolls—80% systematic/equal groups; 7% unsystematic/equal groups
  - Add 3rd doll (redistribute)—74% highly successful

"E", Kindergarten
Justification on
a continuous 2-split

Key distinction: many-to-one vs. many-as-one

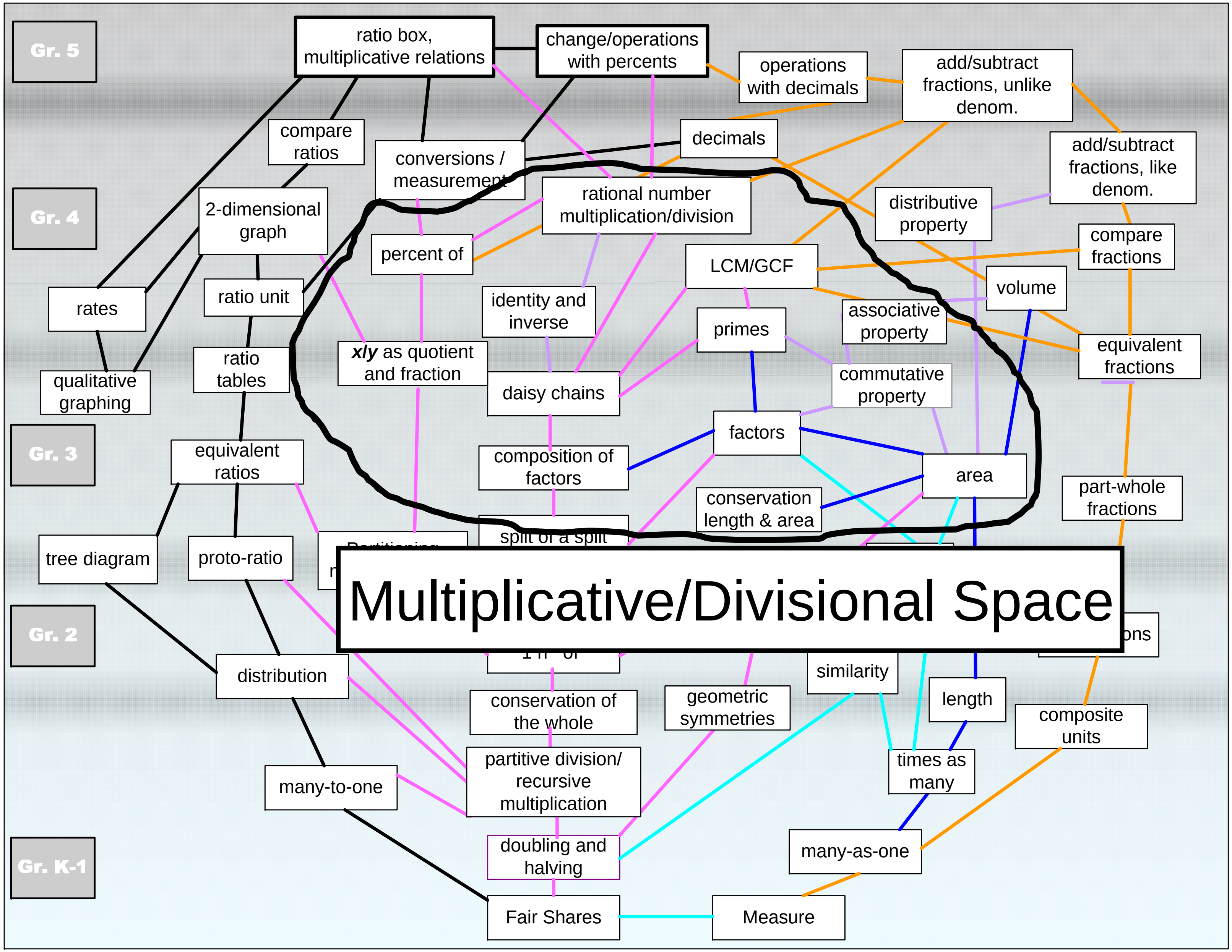


Sharing 20 stars among five children with two types of units marked.

- Many-to-one evolves into unit ratio and unit ratios
- Many-as-one evolves into measures, iterable units and unit fractions

Key Finding 3

- Division and multiplication should be derived from equipartitioning/splitting, and <u>coordinated with</u>, not derived from counting, addition and subtraction;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Fair Shares
Measure

Ex. 2: Counting and Equipartitioning

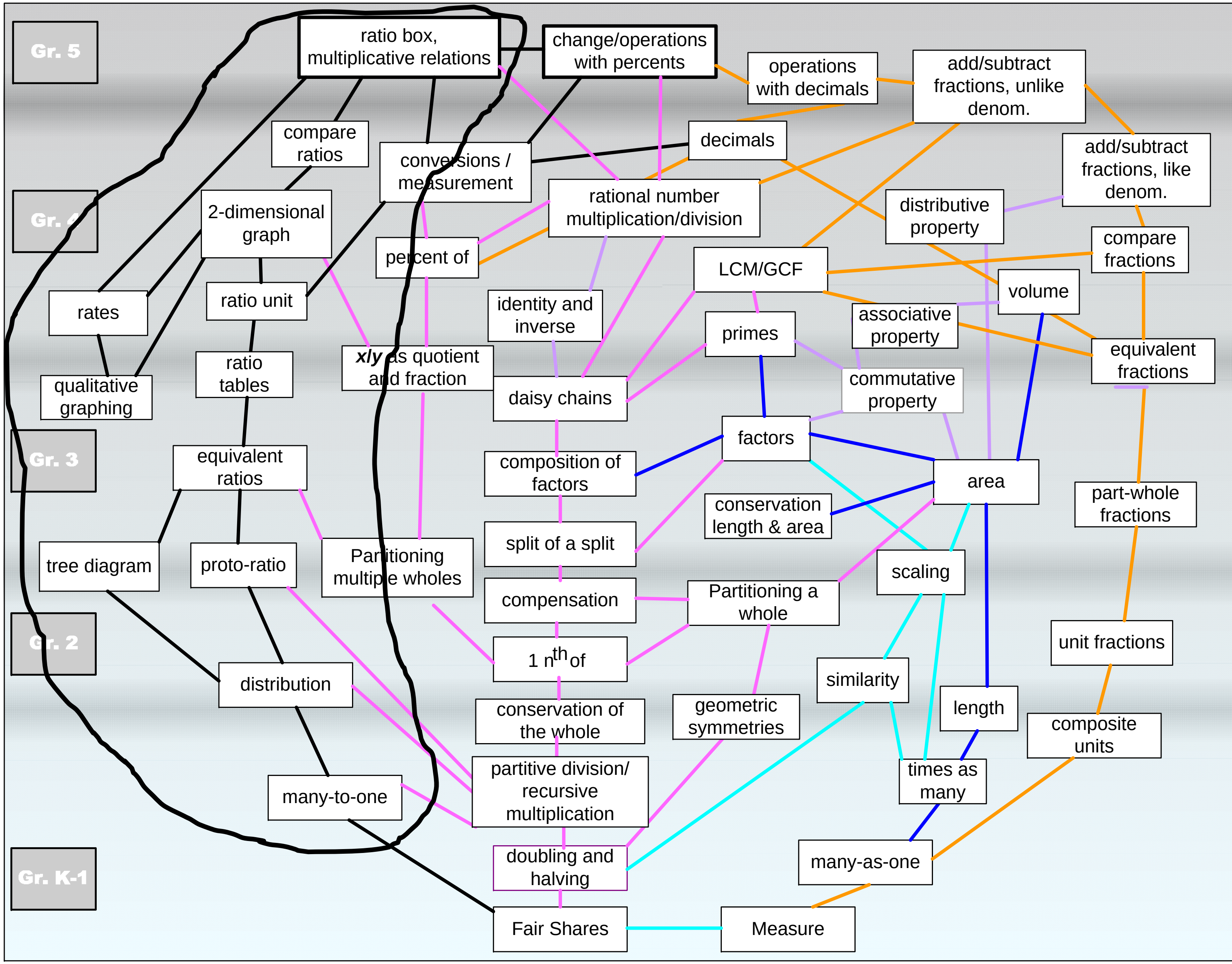
Pepper (1991), Pepper and Hunting (1998): no association between counting and partitioning (4-5 yr-olds)

	Partitioning			
Counting	**Unsystematic, unsuccessful**	**Unsystematic successful**	**Systematic successful**	**Total**
Poor	8	3	34	45 (60.8%)
Developing	1	1	18	20 (27.0%)
Good	1	1	7	9 (12.2%)
Totals	10 (13.5 %)	5 (6.8 %)	59 (79.7%)	74

Key Finding 4: Three fundamental meanings for *a*/*b* capture most of RNR reasoning

- "*a* /*b*" as a relation (ratio, proportion, rate);
- "*a* /*b* of…", for which *a* /*b* is an operator;
- "*a* /*b*" as fraction-as-measure;

Ratio



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“Fair shares” as basis of ratio and proportion

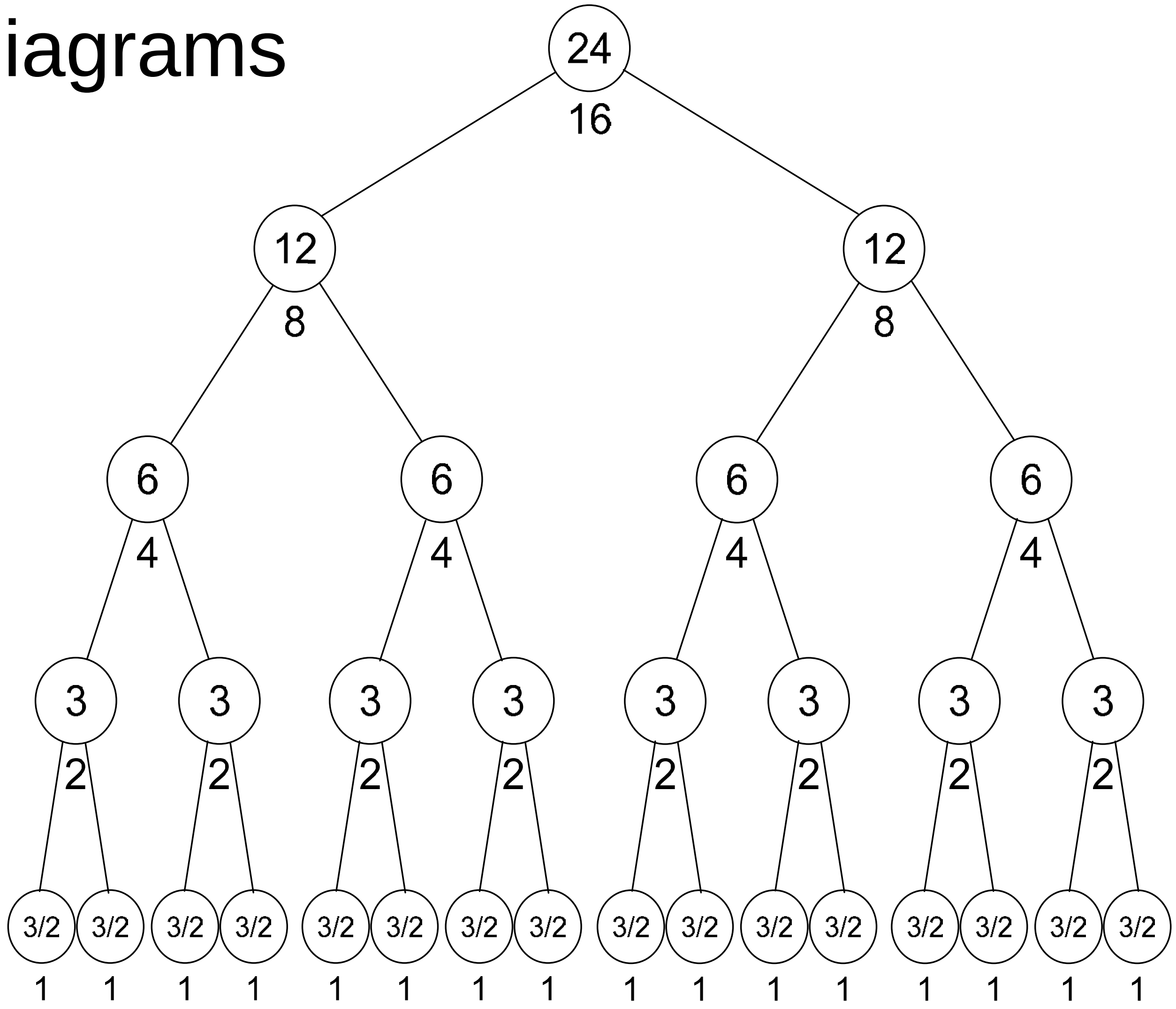
Streefland:

Realistic Mathematics strategies for sharing 24 pizzas among 16 children—but there’s not table big enough for all those children…

How to achieve the same fair shares, at different-sized tables?

Streefland, *Fractions in Realistic Mathematics Education* (1991)

Streefland diagrams



“fair shares” as basis of ratio and proportion

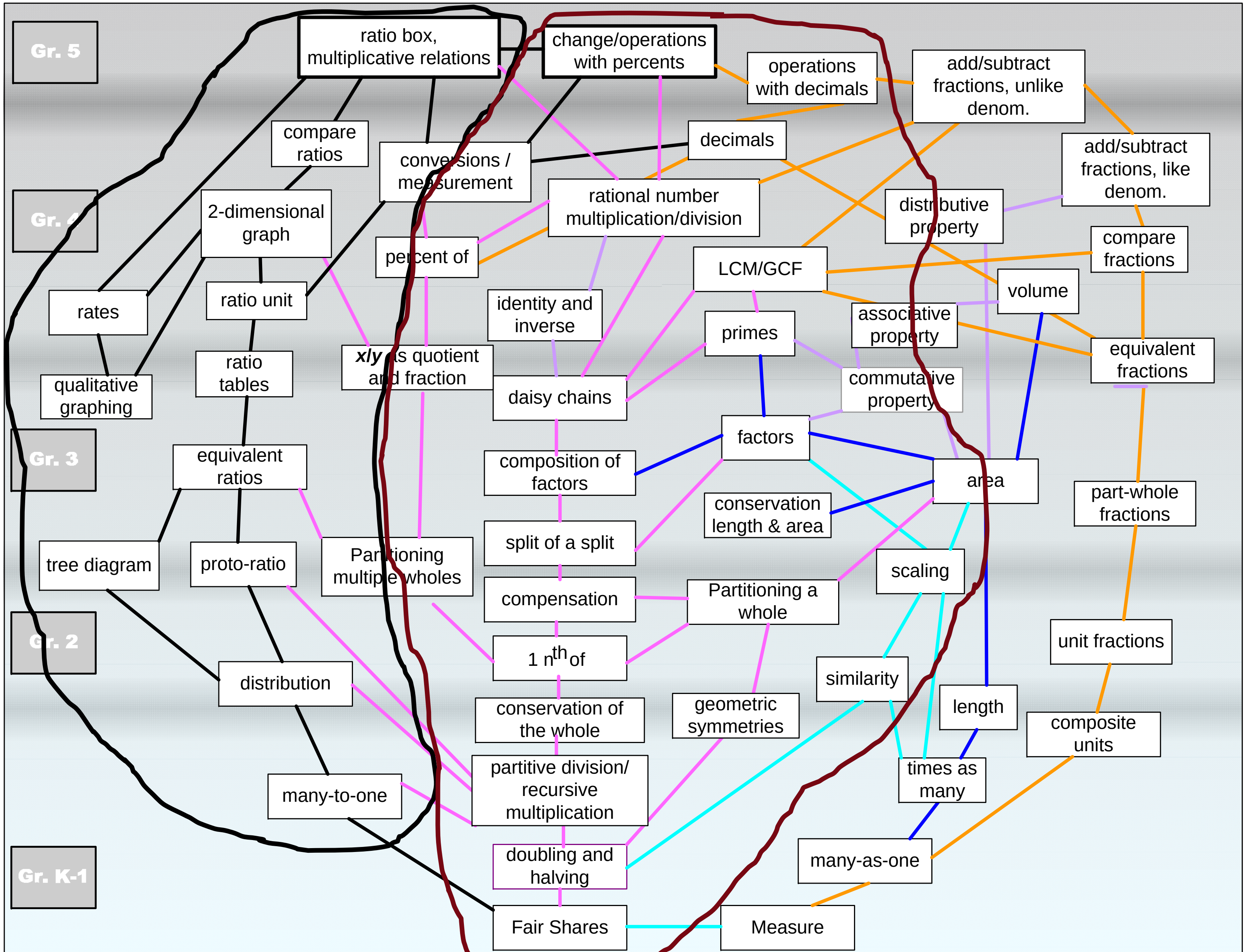
Fit

Equivalence

Covariation

Resnick and Singer (1993)

a/b of…



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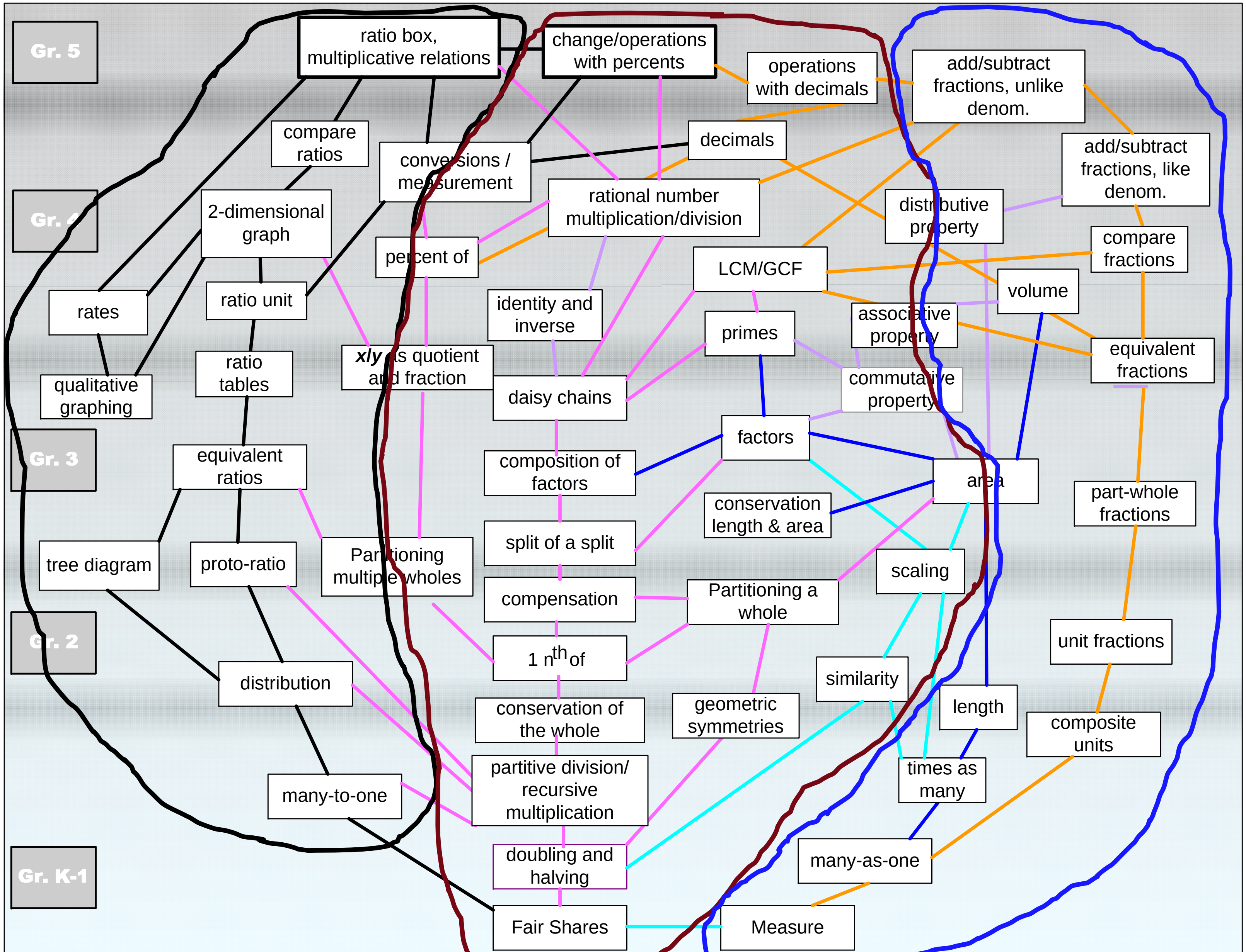
Rational number as operator

- Dienes (*Fractions as Operators*)
- Kieran, Lesh, Behr, Post and Harel (RNP)
- "1/*b*th of" as a primitive (Confrey, 1994)
- "percent of" (Case and Moss, 1998)

a/b as fraction-as-number



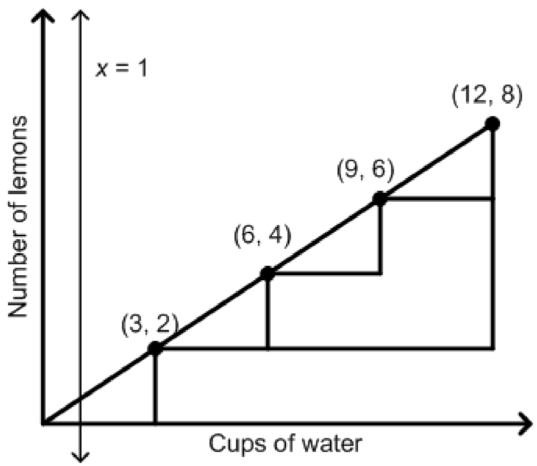
Fraction-as-Number

- Steffe and Olive’s development of fractions: “Children’s fractional schemes can arise as accommodations in their numerical counting schemes.”

Contrasting the Meaning of Equivalence with Ratio and Fraction-as Measure

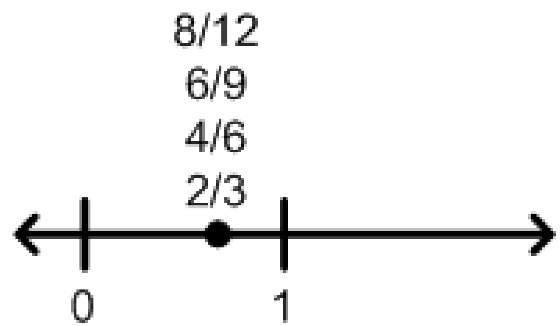
Two dimensional

Cups of water	Number of lemons
3	2
6	4
9	6
12	8



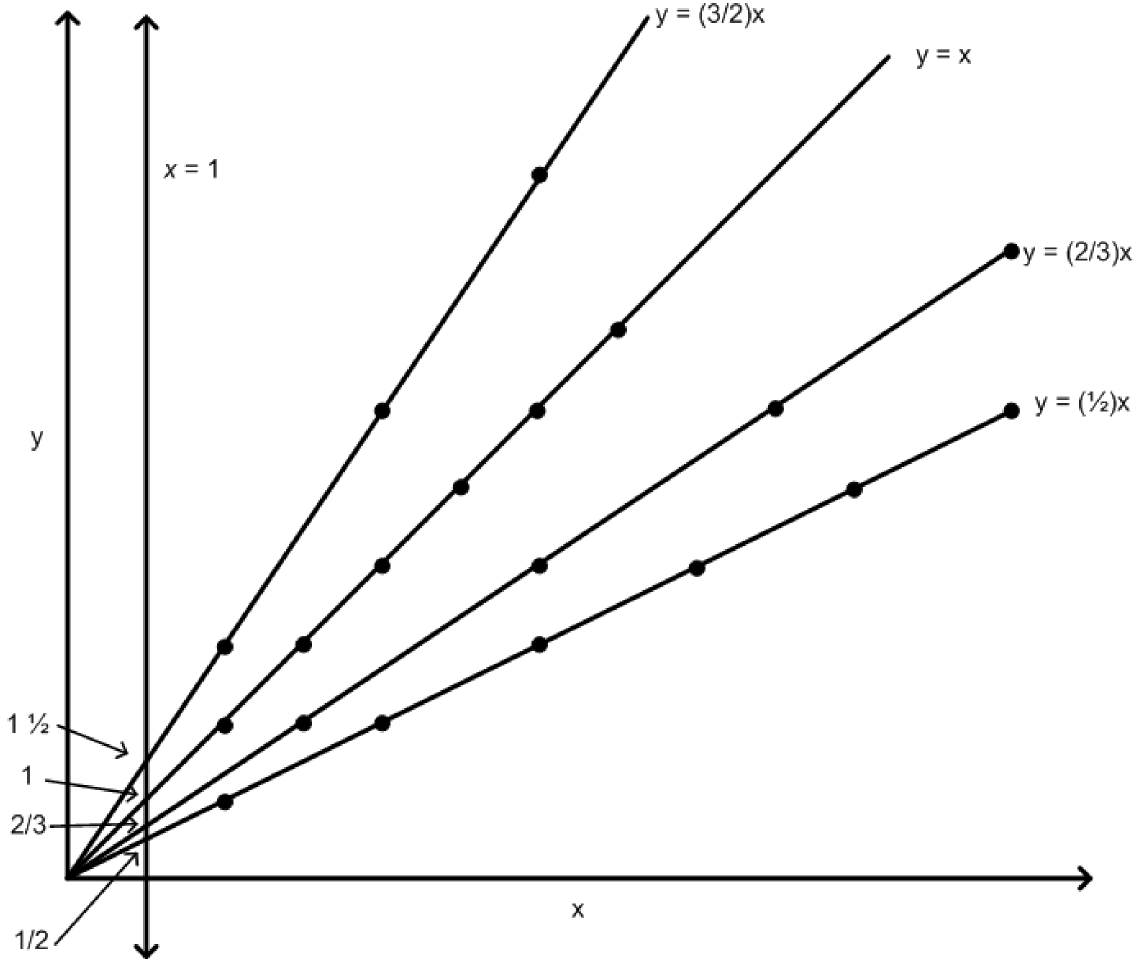
- Equivalence means an invariance in the relationship of two numbers even as the quantities change.

One dimensional



- Fractions have an assumption of a shared unit one.
- Equivalence means to be the same position on number line

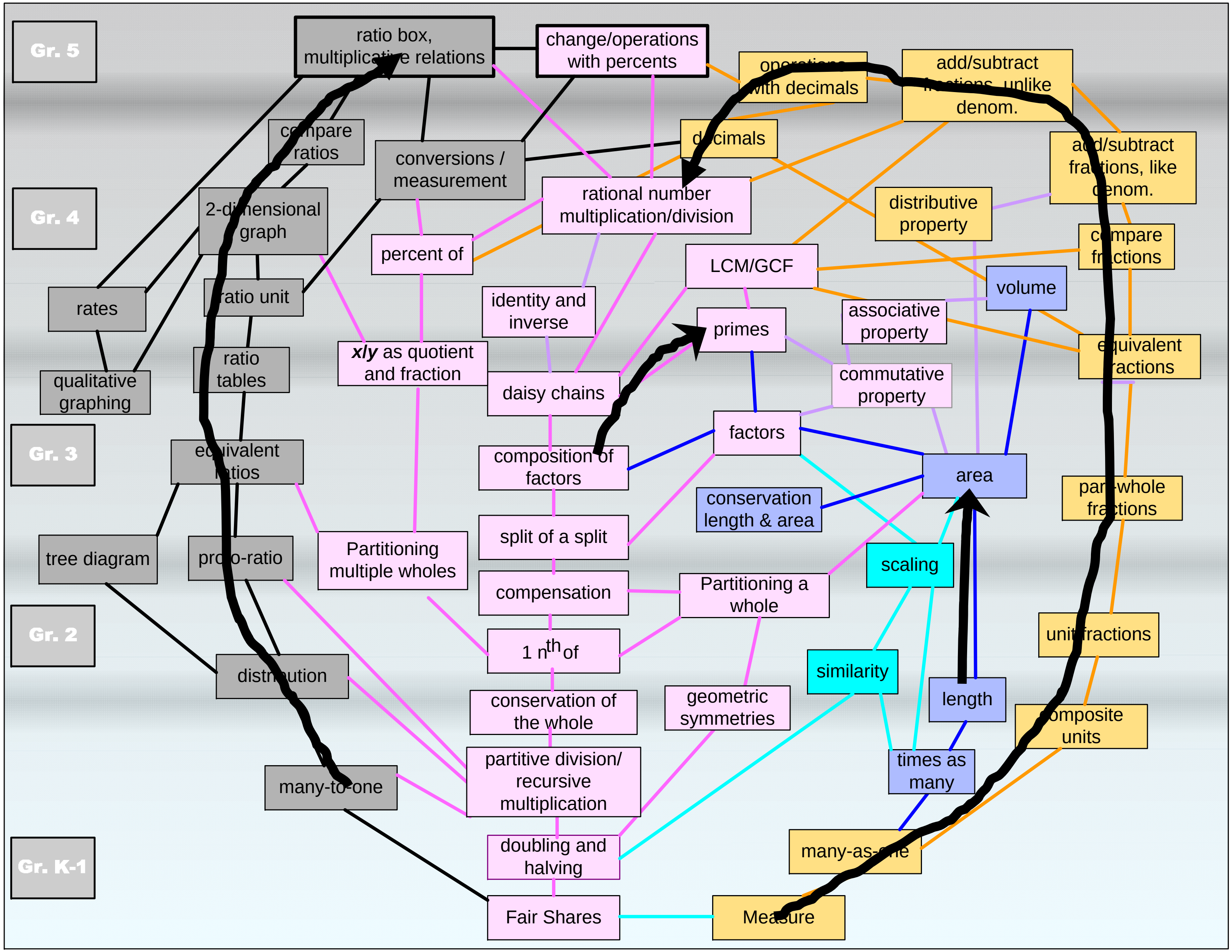
Linking a 1-D and a 2-D representation of *a*/*b*



Key Finding 5:

Fundamental revisions are needed in the sequencing of RNR construct learning.

- Linear Approach [ignores partitioning] :
  1. Multiplication and Division
  2. Area
  3. Fractions (+, -, x, ÷)
  4. Decimals and Percents
  5. Ratios and Proportions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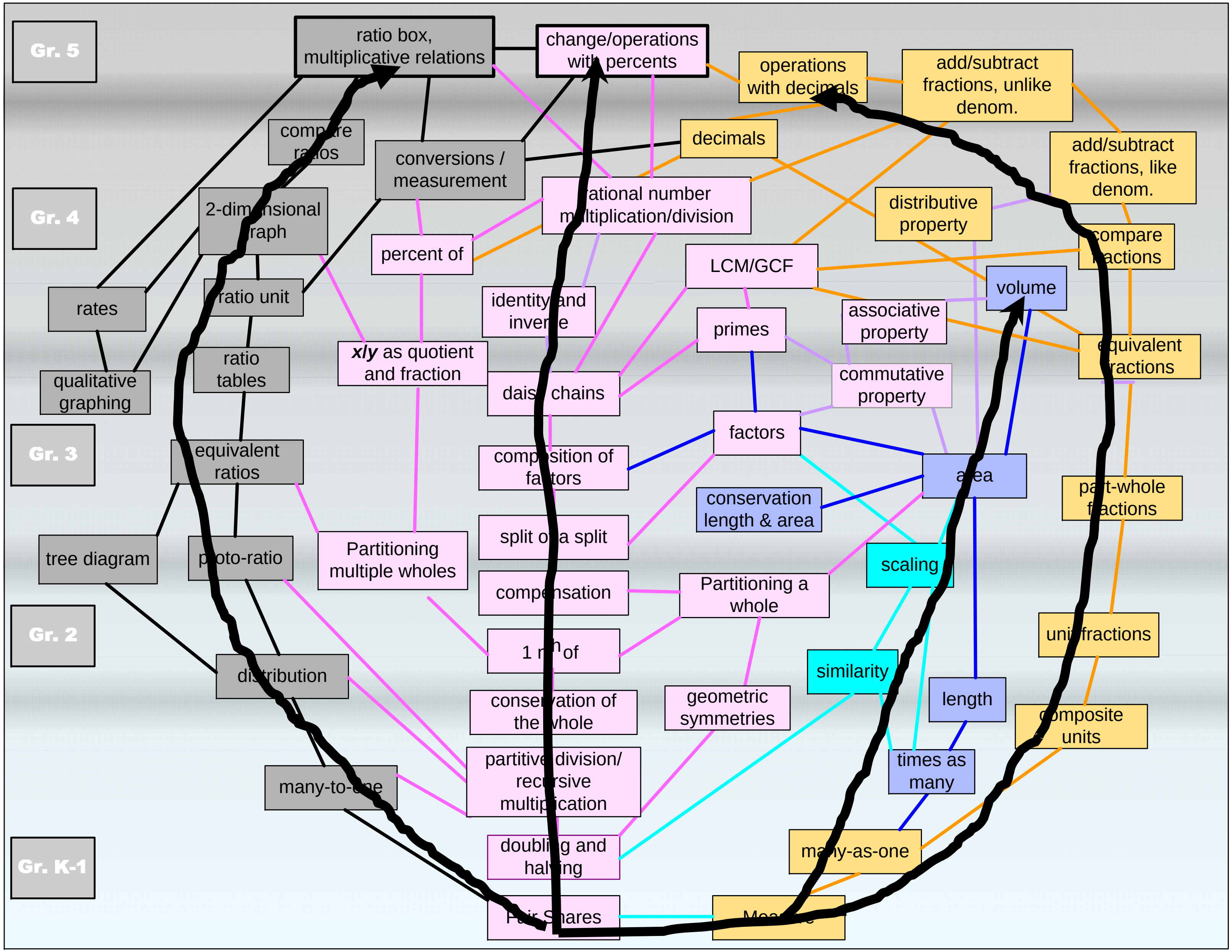
- Parallel Approach:
  - Partitioning to Division and Multiplication
  - Ratio and Proportion
  - Fractions
  - Area and Volume
  - Similarity and Scaling



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Building Diagnostic Measures

Progress Variables (Wilson & Sloane, 2000)

- Focused on progression
- Learning is not just accumulation (i.e. quantitatively more), but qualitatively different
- Ordered levels of understanding
- Derived from professional opinion and empirical research

Partitioning Progress Variables

Case D
Case C
Case B
Case A

1.8 *m* objects shared among *p* people, *m* >*p*
1.7 *m* objects shared among *p* people, *p* >*m*
1.6 Splitting a continuous whole object into odd # of parts (*n* > 3)
1.5 Splitting a continuous whole object among 2*n* people, *n* > 2, and $2n \neq 2^i$
1.4 Splitting continuous whole objects into three parts
1.3 Splitting continuous whole objects into 2^n shares, with *n* > 1
1.2 Dealing discrete items among *p* = 3 - 5 people, with no remainder; *mn* objects, *n* = 3, 4, or 5
1.1 Partitioning using 2-split (continuous and discrete quantities)

Within-level Framework

Properties	Equivalence, Composition, Compensation, Geometric Strategies
Reversibility	“If we put everyone’s share back together, what would it be?”
Naming	“What would you call a share?”
Justification	“How do you know this is a fair share?”
Multiple Methods	“Is there another way to share?”
Methods	“How could you share?”

Consequences of lack of preparation in equipartitioning:

- Video 1 (9’s)
- Video 2 (3’s)

Naïve and misleading conclusions are drawn without adequate study and analysis of children’s reasoning, such as:

- Students’ lack of preparation for algebra is due to their failure to understand fractions.
- Learning fractions depends only on learning arithmetic better.
- Difficulty learning rational numbers will be remedied by treating them exclusively as points on a number line.
- The major challenge in mastering rational number reasoning is the transition from whole numbers to fractions.
- Improving the learning of multiplication and division will be accomplished by better memorization of the facts alone.
- Ratio and proportion should be delayed until middle school.

Conclusions from RNR Synthesis and DELTA

Synthesis indicates that:

- Equipartitioning/splitting is fundamental cognitive root
- Division comes before multiplication from equipartitioning/splitting
- Multiplication and division of rational numbers come before addition and subtraction of rational numbers, and
- "a/b" as ratio, as operator and as fraction-as-measure should be developed in parallel.

Diagnostic Measures should be built on these analyses of student learning, using Progress Variables.

Thank you.